STABILITY OF THE BOUNDARY FLOW SURFACES FORMED BY GRANULAR SOILS AND CRITERION OF DUNE FORMATION

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This paper is concerned with problems of the stability of the beds of rivers and canals. A new method of estimating stability based on dynamic rather than static conditions is proposed; the derivation of this method involves the successive application of the basic concepts of Lyapunov's general theory of stability of motion.

The flow of a liquid in a deformable channel in the presence of scour processes and subsequent filling-up is represented as a wave motion due to the action of perturbations on the "flow-deformable solid boundary" system.

In accordance with the general theory of stability of motion "in the small" and the basic ideas of the hydrodynamic theory of stability [1–3], these perturbations are assumed to be small (in amplitude) sinusoidal waves. If the perturbation has the form of peaks and valleys vanishing at $x = \pm\infty$, it can likewise be expressed in terms of elementary sinusoidal perturbations by means of the Fourier integral [3].

The object of the analysis is to establish the conditions under which the nonsteady regime (perturbed motion) degenerates into the initial uniform state (unperturbed motion), i.e., erosion of the bed and the transport of sediment in the form of waves or dunes once begun decreases exponentially with time.

The representation of the initial phase of deformation of the bed in the form of traveling bottom sand waves is in quite good agreement with the experimental facts [4]. Moreover, it has also been experimentally established that up to certain, relatively large flow velocities the mass motion of sediment along the bottom is wavelike in form.

1. In accordance with the model adopted, the perturbed motion is described by the system of equations of hydraulics (one-dimensional hydrodynamics) of nonstationary motion of a suspensive flow in an erodible channel. This consists of the following three equations:

$$\frac{\partial Q}{\partial x}+\frac{\partial \omega}{\partial t}=q_s, \qquad \frac{\partial Q_s'}{\partial x}+\frac{\partial Q_s''}{\partial x}=-B\frac{\partial Z}{\partial t}, \tag{1.1}$$

$$\frac{1}{g_*}\frac{\partial V}{\partial t}+\frac{\alpha V}{g_*}\frac{\partial V}{\partial x}-\frac{\alpha-1}{g_*}\frac{V}{\omega}\frac{\partial \omega}{\partial t}+\frac{\alpha V}{g_*\omega}q_s+$$
$$+\left(1+\frac{m}{m+2}\sigma S_1\right)\frac{\partial H}{\partial x}+(1+\sigma S_0)\frac{\partial Z}{\partial x}+$$
$$+J^*+\frac{m}{2(m+2)}\sigma H\frac{\partial S_1}{\partial x}=0,$$
$$\sigma=\frac{\rho^*-\rho}{\rho}, \qquad g_*=g\cos\psi, \qquad J^*=\frac{J}{\cos\psi}. \tag{1.2}$$

Here, Q, V, ω, H, and B are, respectively, the discharge, velocity, cross section, depth and width of the flow in the nonstationary regime; Q_s' is the suspended sediment discharge, Q_s'' is the bed load discharge; Z is the height of the bottom of the bed above the horizontal reference plane; α is the total momentum correction, which takes into account both the nonuniformity of the averaged velocity distribution and the velocity fluctuation over the flow cross section; S_0 and S_1 are the mean volume concentration over the cross section and the concentration at the bottom; ρ_* is the density of the sediment; ρ is the density of the water; ψ is the angle at which the bottom is inclined to the horizontal in the stable state; J^* is the hydraulic gradient of the suspensive flow; q_s is the rate of change of discharge, which is governed by the phase influx or efflux along the course.

We will consider the case in which there are no changes in the discharge of the water component, so that q_s expresses the variation of the solid phase or suspended sediment discharge.

The first equation in (1.1) is the continuity equation. The second describes the deformation of the bed, i.e., the transport of the second phase—the material scoured from the bed.

In this form [5] the left-hand side of the equation reflects the variation of the total sediment discharge along the motion, and the right-hand side reflects the scouring or filling-up of the bottom, i.e., the deformation of the bed.

Equation (1.2) is the one-dimensional equation of hydrodynamics (equation of hydraulics) for the nonstationary motion of a turbulent suspensive flow in a deformable channel with variable phase discharge along the course. By analogy with [6] (in which an equation identical with (1.2) was derived for the case of flow in a nondeformable rigid channel), it is obtained from the general equations of hydrodynamics for a two-phase turbulent flow in the Kolmogorov form as proposed by Barenblatt [7]:

$$\frac{\partial U_i}{\partial t}+\frac{\partial U_iU_j}{\partial x_j}=(1+\sigma S)g_i-\frac{1}{\rho}\frac{\partial p}{\partial x_i}-\frac{2}{3}\frac{\partial \sigma}{\partial x_i}+\frac{\partial}{_j\partial x}(\nu\varepsilon_{ij}). \tag{1.3}$$

Equation (1.2) is derived from these as follows: first, the dynamic equation, written in projection onto the normal with respect to the longitudinal component, is integrated over the depth of the flow. The resulting law of distribution of hydrodynamic pressure over the depth of the flow is substituted into Eq. (1.3) written in projection onto the longitudinal axis and then this equation is integrated over the flow cross section. In this case we will confine ourselves to the Boussinesq approximation, discarding as small those terms that contain products of the derivatives and their powers higher than the first. Finally, a power-law approximation is used for the distribution of suspension concentration over the depth. Without presenting proofs, we note that such a relation for the concentration distribution reflects the actual picture at a value of the exponent $m \ll 1$. Thus, we arrive at the equation of hydraulics of sediment-carrying flow in the form (1.2). It immediately goes over into the Boussinesq–Saint-Venant equation if $S_0 = S_1 = 0$.

2. Equations (1.1) and (1.2) include terms containing the suspended and bottom sediment discharges together with the mean and bottom concentrations. Therefore, to close the system, additional relations are required for the above-mentioned characteristics.

In particular, it is possible to use Levi's relation for the discharge [5]. Assuming that $Q \gg Q_s'$, the mean concentration over the flow cross section can be represented in the form*:

$$S_0=Q_s'/Q=0.006\,(V/w_0)^4\,(d/R)^{1.6}. \tag{2.1}$$

Here, w_0 and d are the hydraulic size and diameter of the grains of bed material, R is the hydraulic radius of the channel.

Since S_1 plays an important part in the analysis of the perturbed state of the flow, i.e., the nonstationarity effect, it is not possible to make direct use of existing expressions for S_1 derived for stationary transfer processes in the bottom layer.

However, if we consider that in the presence of bottom perturbations nonstationary pick-up processes may occur if the momentum of the flow in the bottom region, determined from the maximum value of the vertical component of the fluctuation velocity, is greater than the momentum of the grains of the sediments that form the bed, which may be suspended as a result of the action of the perturbations, i.e., greater than the momentum determined from the sinking velocity and

*Instead of (2.1) it would also be possible to take any other relation among those usually employed [8]. They have almost no effect on the end results, as is apparent from the final form of the stability criterion. This is attributable to the fact that at the beginning of the scour processes the mean concentration over the cross section $S_0 \ll S_1$ and in Eq. (1.2) $\sigma S_0 \ll 1$.

the forces of cohesion, then we can write the following approximate formula of phenomenological origin:

$$S_1 = c_0 \frac{W_m^2 - (w_0^2 + T\rho^{-1})}{W_m^2}, \qquad (2.2)$$

where W_m is the maximum value of the vertical component of the fluctuation velocity, T is the cohesive stress, which in the case of silty and clayey particles is introduced to allow for the additional forces that develop in an aqueous medium as a result of intensification of the interaction between the above-mentioned particles, and c_0 is a coefficient that reflects the action on the suspended grains of the field of averaged characteristics of the unperturbed state of the turbulent flow and can be approximately determined from the following considerations.

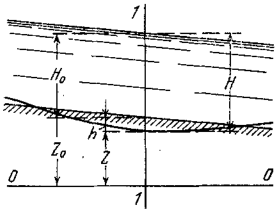

Fig. 1

For a stationary transfer process in the bottom layer Eq. (2.2) can be written in the form:

$$S^*_1 = c_0 \frac{W_m^{*2}(w_0^2 + T\rho^{-1})}{W_m^{*2}}, \qquad (2.3)$$

where W_m^* is W_m referred to the stationary unperturbed state of the flow.

Obviously, the value of S_1^* according to (2.3) should be equal to the existing expressions for S_1 and if (2.3) is equated to one of these, for example, Makkaveev's semiempirical formula [9], which is in good agreement with the experimental data on the bottom concentration under stationary transfer conditions in the bottom layer [10], then for the coefficient c_0 we obtain

$$c_0 = 0.36 \frac{W_m^{*2}}{W_m^{*2} - (w_0^2 + T\rho^{-1})} \frac{V_1^2}{\sigma g_* H_0}$$

$$\left(W_m = 3V_*, \ V_* = \sqrt{g \frac{V_0}{C}}\right). \qquad (2.4)$$

Here, V_* is the so-called dynamic velocity, and C is Chezy's coefficient. Since the bottom velocity V_1 can be expressed in terms of the mean flow velocity in the form $V_1 = V_0 K$, we obtain

$$c_0 = \frac{3.2\, V_0^4 K^2}{\sigma H_0 [9 g_* V_0^2 - C^2 (w_0^2 + T\rho^{-1})]}. \qquad (2.5)$$

3. In the case of concentration of the perturbations essentially within the bottom region of a granular bed, the variation of the free surface owing to the occurrence of perturbations may be neglected; this is equivalent to equating the flow depth increment h to the bottom deformation increment ζ, i.e.,

$$h = H - H_0, \quad \zeta = Z - Z_0, \quad h = -\zeta. \qquad (3.1)$$

Here, H_0 and Z_0 are the flow depth and the vertical position of the channel bottom in the stationary uniform flow regime (Fig. 1), and H and Z are the same quantities in the presence of perturbation, i.e., in the nonstationary regime.

We now linearize Eqs. (1.1) and (1.2), taking

$$V = V_0 + v, \quad H = H_0 + h,$$
$$\omega = \omega_0 + B_0 h, \quad L = L_0 + \zeta,$$

and then, substituting into (1.2) the value of S_1 from (2.2) and the value of q_s from the deformation equation (1.1.2) and eliminating the perturbed velocity v by means of the continuity equation (1.1.1), using condition (3.1), we obtain a second-order partial differential equation with constant coefficients in the deformation coordinate ζ:

$$P_1 \frac{\partial^2 \zeta}{\partial t^2} + P_2 \frac{\partial^2 \zeta}{\partial x \partial t} + P_3 \frac{\partial^2 \zeta}{\partial x^2} + P_4 \frac{\partial \zeta}{\partial t} + P_5 \frac{\partial \zeta}{\partial x} = 0,$$
$$P_1 = 1, \quad P_2 = \alpha V_0 - \alpha A_1^* V_0 B_0 / \omega_0 + A_2 g_*,$$
$$P_3 = \alpha V_0^2 - A_1 \alpha V_0^2 B_0 / \omega_0 - \alpha A_2^* V_0^2 - A_1 g_* \omega_0 / B_0 + A_2 g_* V_0,$$
$$P_4 = 2 g_* i_0 / V_0, \quad P_5 = 2 g_* i_0 \Pi, \qquad (3.2)$$

$$A_1 = \frac{m}{m+2} \sigma c_0 \left[1 - \frac{C^2 (w_0^2 + T\rho^{-1})}{9 g_* V_0^2}\right],$$
$$A_2 = \frac{m}{2(m+2)} \sigma c_0 H_0 \frac{C^2 (w_0^2 + T\rho^{-1})}{9 g_* V_0^3},$$
$$A_1^* = 0.0083\, F_0^{3/2} \left(\frac{\Pi}{x^*}\right)^{3/2} H_0 \left(\frac{H_0}{d}\right)^{1/4} -$$
$$-0.0062\, F_0 \frac{\Pi}{x^*} H_0 \left(\frac{d}{H_0}\right)^{1/12},$$
$$A_2^* = 0.0005\, F_0^{3/2} \left(\frac{\Pi}{x^*}\right)^{3/2} \left(\frac{H_0}{d}\right)^{1/4} - 0.0002\, F_0 \frac{\Pi}{x^*} \left(\frac{d}{H_0}\right)^{1/12},$$
$$\Pi = \frac{\omega_0 x^*}{2 B_0 H_0}, \quad F_0 = \frac{V_0^2}{g_* \omega_0 / B_0}. \qquad (3.3)$$

Here, Π is the so-called form factor of the bed, F_0 is the Froude number, i_0 is the slope of the bottom, and x^* is the hydraulic index of the bed.

The stability of the unperturbed state of the channel bottom is analyzed below by representing the solution of Eq. (3.2) in the form (λ is the wavelength):

$$\zeta = e^{\beta x + rt} \qquad (3.4)$$

with $\mathrm{Re}\,\beta = 0$ in order to satisfy the conditions at infinity [1].

Then the characteristic equation obtained by introducing values of ζ from (3.4) into Eq. (3.2) has the form:

$$P_1 r^2 + (P_4 + ikP_2) r + (-P_3 k^2 + ikP_5) = 0$$
$$(k = \mathrm{Im}\,\beta = 2\pi/\lambda). \qquad (3.5)$$

We will employ the Lyapunov-Hurwitz stability condition used in [12] to investigate the stability of a real fluid flow of finite depth in

d, mm	H, m	U, m/sec	V_0^*, m/sec	d, mm	H, m	U, m/sec	V_0^*, m/sec
1.0	0.4	0.47	0.42	15.0	0.4	0.95	0.97
	1.0	0.57	0.49		1.0	1.20	1.15
	2.0	0.65	0.56		2.0	1.30	1.17
	3.0	0.70	0.62		3.0	1.40	1.40
5.0	0.4	0.65	0.70	25.0	0.4	1.20	1.16
	1.0	0.80	0.82		1.0	1.40	1.40
	2.0	0.90	0.93		2.0	1.60	1.56
	3.0	0.95	1.01		3.0	1.80	1.72
10.0	0.4	0.80	0.94	40.0	0.4	1.50	1.40
	1.0	1.00	1.03		1.0	1.80	1.89
	2.0	1.10	1.17		2.0	2.10	1.98
	3.0	1.20	1.24		3.0	2.20	2.13

nondeformable channels. According to this condition, the motion is asymptotically stable if all the even minors in the left upper corner of the square matrix of order 2n composed of coefficients of the polynomial $f(iZ)$ representing the characteristic equation with the complex coefficients of the equation of perturbed motion are greater than zero.

From the square matrix constructed from (3.5) (by reducing it to the form $if(iZ)$) we obtain the following stability conditions:

$$P_1P_4 > 0, \quad P_2P_4P_5 - P_3P_4^2 - P_1P_5^2 > 0 \;,$$
$$(P_1P_4 = 2g_* i_0 / V_0) > 0 \,. \tag{3.6}$$

The first of these is always satisfied in accordance with the expression in parentheses.

Introducing into the second condition the value of the constant coefficients according to (3.3), neglecting the terms with cofactors A_1^* and A_2^* as negligibly small, and solving the inequality for the flow velocity, we obtain

$$V_0^* < \Big[\Big(\delta\, C^2(w_0^2 + T\rho^{-1})\,\{G - 0.36\, m_1\, K^2\omega_0 B_0^{-1} \times$$
$$\times [1 - 0.25\, x^* (1 - \Pi^{-1})]\}\Big) \times$$
$$\times \Big(9g^* (G - 0.36\, m_1\, K^2\omega_0 B_0^{-1})\Big)^{-1}\Big]^{1/2}$$
$$G = H_0 (\Pi^2 - \alpha\Pi + \alpha). \tag{3.7}$$

The right-hand side of (3.7) expresses the limiting velocities above which the stable state of the channel must be disturbed, i.e., dune formation begins.

These velocities are directly related with the diameter of the bed material—the greater the diameter, the greater the right-hand side of inequality (3.7), i.e., the degree of stability of the bed increases. Thus, the criterion gives a qualitatively correct picture of the actual process of erosion and filling-up of the beds of rivers and canals. Quantitatively, it is possible to obtain an estimate by comparing the velocities calculated from (3.7) with the values of the so-called nonscouring velocities given in existing standards [13], which are the result of generalizing the empirical relations for nonscouring velocities, the results of laboratory and field experiments, and operational data. Since the velocities given in the standards chiefly correspond to rectangular and broad channels, the values of the stable velocities calculated from (3.7) are given in the table for the same section.

As may be seen from the table, $V_0^* \approx U$, i.e., there is very satisfactory agreement between the theory and the generalized material of an experimental nature. This is all the more significant in that not one empirical coefficient has been introduced into the criterion obtained in order to improve the convergence of the results and the criterion is composed exclusively of perfectly definite flow and channel characteristics.

The degree of quantitative convergence is graphically illustrated in Fig. 2, where V_1 is the effective velocity (m/sec) at which the grain starts to move and d is the grain diameter (mm); curves 1–4 are based on relation (3.7) with the Chezy coefficient according to Makkaveev (1), Tou Kuo-jen (2), Ch'ang (3), and Strickler (4); the experimental points are those of: Velikanov—1; Pushkarev—2; Shamov—3; Knoroz—4; Rubinshtein—5; He—6; Chang—7; Nanking Laboratory—8; field observations of Chinese rivers—9; Meyer-Peter—10; Scobey—11; and Shaffernak—12. The comparative experimental points were taken from [14], therefore the conversion from the velocities obtained from (3.7) to V_1 was based on Tou Kuo-jen's relation for the distribution of local averaged velocity along the normal to the free surface [15].

On the graph we have plotted the experimental points obtained by M. A. Velikanov and N. M. Bochkov, V. F. Pushkarev, G. I. Shamov, V. S. Knoroz, G. A. Rubinshtein, He Chih-t'a, Chang Yu-ling, the Nanking Hydraulic Engineering Laboratory, Meyer-Peter, Scobey, and Shaffernak [14]. Four theoretical curves are also shown. They were all obtained from (3.7), but using different relations for the Chezy coefficient C, namely, the expressions proposed by Strickler, Ch'ang, V.M. Makkaveev [16], and Tou Kuo-jen [15].

In constructing the theoretical curves we introduced these expressions for C because in them the Chezy coefficients are quantitatively related with the bed grain diameter, which is very important. Since these expressions do not give the same value of C, we obtained four theoretical curves. However, all these curves conform quite well to the experimental points. This again shows that the stability criterion also gives quantitatively reliable and correct results.

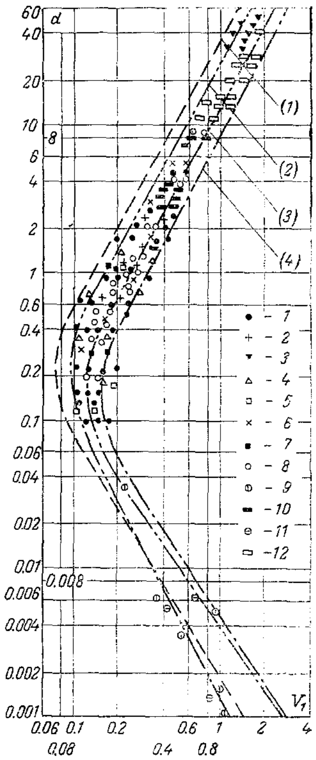

Fig. 2

The theoretical curves have a minimum corresponding to the least values of the velocities at a grain diameter of approximately 0.2 mm. These fractions are the most "dangerous" in relation to displacement. The existence of "dangerous" fractions and the corresponding minimum critical velocities has also been noted by experimeters such as Scobey and Fortier, Penk, Schoklitsch, Gilbert, Shaffernak, and others [17].

To each value of the velocity above the "critical" there correspond "dangerous" fractions of two different diameters. For example, $V_1 = 0.5$ m/sec, grains belonging to the fraction from 0.01 to 5 mm are unstable. Grains smaller than 0.01 and larger than 5 mm will not be transported at that velocity. This is because at diameters <0.1–0.2 mm the cohesion between the grains and hence their resistance to stripping and shear increase sharply. This is confirmed by observation and experiment [18, 19].

It should be noted that similar, but exclusively empirical curves have been obtained by a number of investigators, for example, Bagnold, Sundborn, Zvonkov [17], and Tou Kuo-jen [14].

In conclusion we note that in deriving the theoretical $V_1 = f(d)$ curves, we took values of 2.5 for x^* in (3.7). This most closely corresponds to the conditions under which the experimental comparison points were obtained.

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REFERENCES

1. Chia-ch'iao Lin, Hydrodynamic Stability [Russian translation], IL, 1958.

2. Shih-i Pai, Theory of Jets [Russian translation], Fizmatgiz, 1960.

3. Chia-shun I, "Stability of a liquid flow along an inclined plane," collection: Mekhanika, 1963.

4. V. N. Goncharov, Fundamentals of the Dynamics of Channel Flows [in Russian], Gidrometeoizdat, 1954.

5. I. I. Levi, Dynamics of Channel Flows [in Russian], Gosenergoizdat, 1957.

6. O. G. Natishvili, "Stability of motion of a sediment-carrying turbulent flow at high velocities in channels with a steep slope," Tr. Gruz. NIIGim, no. 22, 1963.

7. G. I. Barenblatt, "Motion of suspended particles in a flow," PMM, vol. 17, no. 3, 1953.

8. I. V. Egiazarov, "Generalized transport equation for noncohesive sediments, resistance coefficient of an erodible channel and the nonscouring velocity," Proc. 3rd All-Union Hydrolog. Conf. [in Russian], vol. 5, 1960.

9. V. M. Makkaveev and I. M. Konovalov, Hydraulics [in Russian], Rechizdat, 1940.

10. I. I. Levi, "Slowly varying motion of a liquid in erodible channels with arbitrary longitudinal bottom profile," Izv. VNIIG, vol. 30, 1941.

11. A. V. Karaushev, Problems of the Dynamics of Natural Water Flows [in Russian], Gidrometeoizdat, 1960.

12. T. G. Voinich-Syanozhentskii, "Problems of the stability of motion of a real-liquid flow in channels of finite depth," Izv. TNISGEI, vol. 16, 1965.

13. P. G. Kiselev, Manual of Hydraulic Calculations [in Russian], Gosenergoizdat, 1957.

14. Kuo-jen Tou, "Problems of the stability of river beds," Proc. 3rd All-Union Hydrolog. Conf. [in Russian], vol. 5, 1960.

15. V. M. Makkaveev, "Processes of formation of rotational fluctuation motions at the boundary surfaces of a turbulent flow," Tr. LIVT, no. 77, 1964.

16. V. M. Makkaveev, "Longitudinal and transverse velocity distribution in open flows," Tr. GGI, 2 (56), 1947.

17. V. V. Zvonkov, Water and Wind Erosion [in Russian], Izd-vo AN SSSR, 1963.

18. S. V. Nerpin, A. I. Kotov, and D. N. Rosha, Foundations and Engineering Geology [in Russian], Rechizdat, 1963.

19. G. S. Bashkirov, The Dynamics of Near-Shore Waters [in Russian], Izd. Morskoi transport, 1961.

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